About one inequality. Its proofs and short history

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We start from inequality posted in MOG (Mathematical Olympiad Group) by Elton Papanikolla (Group Owner) at 02.26.2008, link: https://www.linkedin.com/groups/8313943/8313943-6374143272943783937.

Namely, let a,b,c be positive real numbers. Show that

(**1**) $a+b+c+3\sqrt[3]{abc} \geq 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$.

Together with inequality (**1**) represented in this problem consider three more inequalities

(**2**) $x^6y^6+y^6z^6+z^6x^6+3x^4y^4z^4 \geq 2(x^3+y^3+z^3)x^3y^3z^3$ where $x,y,z \in \mathbb{R}$;

(**3**) $x^6+y^6+z^6+3x^2y^2z^2 \geq 2(x^3y^3+y^3z^3+z^3x^3)$ where $x,y,z \in \mathbb{R}$;

(**4**) $(x^3+y^3+z^3)^2+3(xyz)^2 \geq 4(x^3y^3+y^3z^3+z^3x^3)$, where $x,y,z \in \mathbb{R}$.

(Inequalities (**2**) and (**3**) are represented in M1834, KVANT n.5, 2002, proposed by F. Shleyfer and inequality (**4**) represented in Problem 2839 CRUX vol.29,n.4, 2003, proposed by M. Klamkin).

First note, that inequalities in (**2**) and (**3**) are equivalent. Really, by replacing (x,y,z) in inequality (**2**) with $\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ we obtain

$$\frac{1}{x^6y^6}+\frac{1}{y^6z^6}+\frac{1}{z^6x^6}+\frac{3}{x^4y^4z^4} \geq 2\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)\frac{1}{x^3y^3z^3} \Leftrightarrow (\mathbf{3})$$

Also easy to see that inequality (**4**) is essentially inequality (**3**) .

And, at last, by replacing (x,y,z) in (**3**) with $\left(\sqrt[6]{a},\sqrt[6]{b},\sqrt[6]{c}\right)$ (in case $x,y,z>0$) we obtain inequality (**1**). Thus, we can see that all these inequalities up to notation are the same in case of positivity of variables.

But for the my proof that follows is more convenient another one equivalent form of these inequalities, namely by replacing (x,y,z) in (**3**) with $\left(\sqrt[3]{a},\sqrt[3]{b},\sqrt[3]{c}\right)$ we obtain inequality

(**5**) $a^2+b^2+c^2+3\sqrt[3]{a^2b^2c^2} \geq 2(ab+bc+ca)$, where $a,b,c \in \mathbb{R}$.

Without loss of generality we can for further suppose that $a,b,c>0$ because if one of a,b,c is 0 inequality became obvious and $|ab|+|bc|+|ca| \geq ab+bc+ca$.

Multiplying inequality $3\sqrt[3]{abc} \leq a+b+c$ by $3\sqrt[3]{a^2b^2c^2}$ we obtain inequality $9abc \leq 3\sqrt[3]{a^2b^2c^2}\left(a+b+c\right)$ which in combination with Schur's inequality $\sum a(a-b)(a-c) \geq 0$

in form $4(ab+bc+ca)(a+b+c)-(a+b+c)^3 \leq 9abc$ gives us inequality

$$3\sqrt[3]{a^2b^2c^2}\left(a+b+c\right) \geq 4(ab+bc+ca)(a+b+c)-(a+b+c)^3 \Leftrightarrow$$

$$3\sqrt[3]{a^2b^2c^2} \geq 4(ab+bc+ca)-(a+b+c)^2 \Leftrightarrow (\mathbf{5}).$$

Thus, we also proved inequalities (**4**),(**3**),(**2**), (**1**)

This proof of inequality (**5**) is at the same time the shortest proof of inequalities (**4**),(**3**),(**2**)) among all others given in the above sources [2], [3].

Second proof.

Using normalization by $a+b+c=1$ (due homogeneity of (**5**)) and denoting

$p := ab + bc + ca$,

$q := abc$ we can rewrite inequality (5) as follows

$1 - 2p + 3\sqrt[3]{q^2} \ge 2p \Leftrightarrow 3\sqrt[3]{q^2} \ge 4p - 1 \Leftrightarrow 27q^2 \ge (4p-1)^3$.

Since $q \ge \frac{4p-1}{9}$ ($\Leftrightarrow \sum a(a-b)(a-c) \ge 0$ – Schur's Inequality) and $p \le \frac{1}{3}$ ($\Leftrightarrow 3(ab+bc+ca) \le (a+b+c)^2$) then

$27q^2 - (4p-1)^3 \ge 27 \cdot \left(\frac{4p-1}{9}\right)^2 - (4p-1)^3 = (4p-1)^2\left(\frac{1}{3} - (4p-1)\right) =$

$4(4p-1)^2\left(\frac{1}{3} - p\right) \ge 0$.

Remark(**more then one proof**).

Inequality (**5**) can be rewritten in form

$$\boxed{3\sqrt[3]{a^2b^2c^2} \ge \Delta(a,b,c)} \qquad [1]$$

where $\Delta(a,b,c) = 2ab + 2bc + 2ca - a^2 - b^2 - c^2$.

Then by Schur's Inequality $\sum_{cyc} a(a-b)(a-c) \ge 0$ in form

$$\boxed{9abc \ge (a+b+c)\Delta(a,b,c)}$$

and AM-GM inequality $a + b + c \ge 3\sqrt[3]{abc}$ we have

$$3\sqrt[3]{a^2b^2c^2} - \Delta(a,b,c) = \frac{9abc - 3\sqrt[3]{abc}\,\Delta(a,b,c)}{3\sqrt[3]{abc}} =$$

$$\frac{9abc - (a+b+c)\Delta(a,b,c) + \Delta(a,b,c)\left(a+b+c-3\sqrt[3]{abc}\right)}{3\sqrt[3]{abc}} \ge 0.$$

1.**Arkady Alt**-"**Geometric Inequalities with polynomial 2xy**+**2yz**+**2zx2**-**sqr**(**x**)-**sqr**(**y**)-**sqr**(**z**)"

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3. 2839 Solution, **CRUX v.30**,**n.4**

Link: https://cms.math.ca/crux/v30/n4/

Remark. The proofs represented here I found at 2009 and all of them in the file "Collection of my shortest proofs of hard inequalities."

Bravo! Giovanni Parzanese. I was pleased to see your proof of the inequality because it by essense coinside with one of my proofs of the inequality

$x^6 + y^6 + z^6 + 3x^2y^2z^2 \ge 2(x^3y^3 + y^3z^3 + z^3x^3)$ where $x, y, z \in \mathbb{R}$

presented in the problem M1834, KVANT n.5, 2002, proposed by F. Shleyfer and which up to notation is the same as inequality

$a + b + c + 3\sqrt[3]{abc} \ge 2\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right)$.

Idea of combination AM-GM and Schur's inequalities give the shortest proof among others published. The details by link presented above.